Tutorial 6

In this tutorial, we review some subtle concepts which may be helpful for preparing for the midterm.

1. Recall from class that if $f(z)$ is analytic at some point z_0, then its component functions will necessarily satisfy the Cauchy-Riemann equations at that point. However, the following example demonstrates that the converse is not always true.

If the function $f(z) = u(x, y) + iv(x, y)$ is defined by

$$f(z) = \begin{cases} \bar{z}^2/z & z \neq 0, \\ 0 & z = 0, \end{cases}$$

its real and imaginary parts are

$$u(x, y) = \frac{x^3 - 3xy^2}{x^2 + y^2} \quad \text{and} \quad v(x, y) = \frac{y^3 - 3x^2y}{x^2 + y^2}$$

when $(x, y) \neq (0, 0)$. Also, $u(0, 0) = v(0, 0) = 0$.

The Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ are satisfied at $z = 0$, but $f'(0)$ does not exist.

Note. The component functions are not continuous at $(0, 0)$.

2. Sometimes when considering an integral over a simple closed curve, you might apply the Cauchy-Goursat theorem carelessly. The following example shows that it is always important to check if the function satisfies the hypothesis in the theorem.

Let C be the positively oriented circle of radius 1 centered at $z = -4$ and let $f(z) = 1/(z+4)$. Consider the following integral

$$I = \int_C f(z)\ dz.$$

Note that f is not analytic at the center $z = -4$, so we cannot apply the Cauchy-Grousat theorem. Instead, we parametrize the circle by $z = -4 + e^{i\theta}$ with $0 \leq \theta \leq 2\pi$ and so

$$I = \int_C \frac{1}{z+4}\ dz = \int_0^{2\pi} \frac{1}{e^{i\theta}} ie^{i\theta}\ d\theta = \int_0^{2\pi} i\ d\theta = 2\pi i.$$